# Introduction to Multiple Linear Regression

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*This material is part of the* ***statsTeachR*** *project*

# Today's lecture

## Multiple Linear Regression: basic concepts

- Motivation
- Assumptions
- Interpretation of $\beta$s
- More on confounding (omitted variable bias)
- Matrix notation for MLR

Relevant reading: Faraway Chapter 2, *ISL* Chapter 3.2-3.3

# Motivation

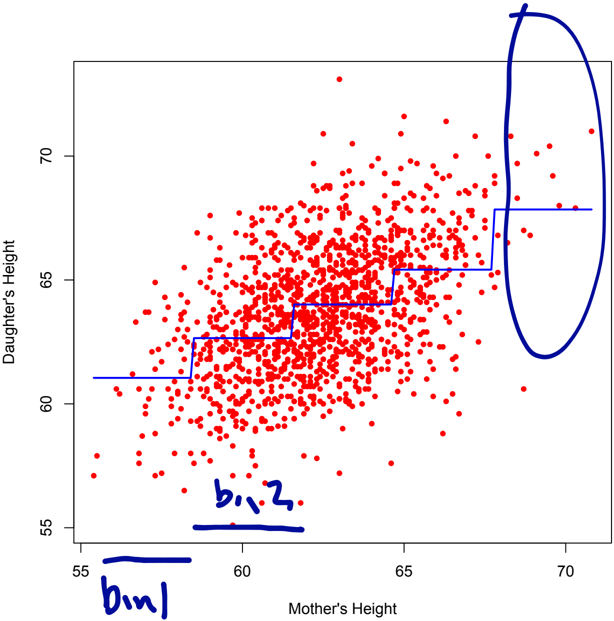
Most applications involve more that one covariate – if more than one thing can influence an outcome, you need multiple linear regression.

- Improved description of $y|\mathbf{x}$ boldface
- More accurate estimates and predictions
- Allow testing of multiple effects
- Includes multiple predictor types

# Why not bin all predictors?

- Divide $x_i$ into $k_i$ bins
- Stratify data based on inclusion in bins across $x$'s
- Find mean of the $y_i$ in each category
- Possibly a reasonable non-parametric model

# Why not bin all predictors?



# Why not bin all predictors?

- More predictors = more bins
- If each $x$ has 5 bins, you have $5^p$ overall categories
- May not have enough data to estimate distribution in each category
- Curse of dimensionality is a problem in a lot of non-parametric statistics

For more, see this interactive Shiny app.

# Multiple linear regression model

- Observe data $(y_i, x_{i1}, \ldots, x_{ip})$ for subjects $1, \ldots, n$. Want to estimate $\beta_0, \beta_1, \ldots, \beta_p$ in the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_1 x_{ip} + \epsilon_i; \; \underline{\epsilon_i \overset{iid}{\sim} (0, \sigma^2)}$$

- Assumptions (residuals have mean zero, constant variance, are independent) are as in SLR
- Impose linearity which (as in the SLR) is a big assumption
- Our primary interest will be $E(y|\mathbf{x})$ ← vector
- Eventually estimate model parameters using least squares

# Predictor types

- Continuous
- Categorical
- Ordinal

binary

category, unordered
e.g race
"nominal"

categorical
e.g. high school
grade level

# Interpretation of coefficients

$\beta_0 = E(y|x_1 = 0, \ldots, x = 0)$

- Centering some of the $x$'s may make this more interpretable

$$E(y|x) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$

All x's must have 'meaningful' zero value for $\beta_0$ to have good interpretation.

# Interpretation of $\beta_1$

$$E(y \mid x_1 = k_1, x_2 = k_2, x_3 = k_3) = \cancel{\beta_0} + \cancel{\beta_1 k_1} + \cancel{\beta_2 k_2} + \cancel{\beta_3 k_3}$$

$$- E(y \mid x_1 = k_1 - 1, x_2 = k_2, x_3 = k_3) = \cancel{\beta_0} + \cancel{\beta_1 k_1} - \beta_1 + \cancel{\beta_2 k_2} + \cancel{\beta_3 k_3}$$

change in $E(y|x)$ for a 1 unit ↑ in $x_1$, all other $x$'s held constant $= \beta_1$

$x_2$ $\qquad\qquad$ $\beta_2$

# Example with two predictors

Suppose we want to regress weight y on ~~height~~ age and sex.

- Model is $y_i = \beta_0 + \beta_1 x_{i,age} + \beta_2 x_{i,sex} + \epsilon_i$
- Age is continuous starting with age 0; sex is binary, coded so that $x_{i,sex} = 0$ for men and $x_{i,sex} = 1$ for women

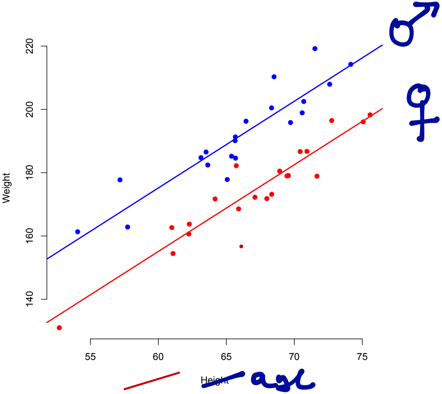
# Example with two predictors

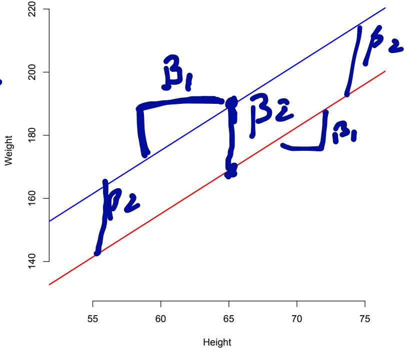
Model: $y_i = \beta_0 + \beta_1 x_{i,age} + \beta_2 x_{i,sex} + \epsilon_i$

$\beta_1 =$ the expected change in weight for a 1 unit increase in age for people of the same gender/sex

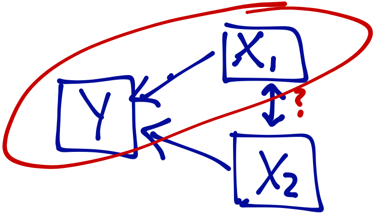
$\beta_2 =$ the difference in expected weight between a man and a woman of the same age

# Example with two predictors



# Omitted variable bias




What happens if the true regression model is

$$y_i = \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \epsilon_i$$

but we ignore $x_2$ and fit the simple linear regression

$$y_i = \beta_0^* + \beta_1^* x_{i,1} + \epsilon_i^*$$

Does $\beta_1^* = \beta_1$?

# Omitted variable bias

## When should you be concerned?

If both of the following conditions are met, then $\beta_1^* = \beta_1$:

- The omitted variable is unrelated to the outcome
- The omitted variable is uncorrelated with the retained variable

Note: A Simpson's paradox can be explained by ommited variable bias.

# Matrix notation

- Observe data $(y_i, x_{i1}, \ldots, x_{ip})$ for subjects $1, \ldots, n$. Want to estimate $\beta_0, \beta_1, \ldots, \beta_p$ in the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_1 x_{ip} + \epsilon_i;\ \epsilon_i \overset{iid}{\sim} (0, \sigma^2)$$

- Notation is cumbersome. To fix this, let
  - $\mathbf{x}_i = [1, x_{i1}, \ldots, x_{ip}]$ $1 \times p$
  - $\boldsymbol{\beta}^T = [\beta_0, \beta_1, \ldots, \boldsymbol{\beta}_p]$ $\beta_{p \times 1}$
  - Then $y_i = \mathbf{x}_i \boldsymbol{\beta} + \epsilon_i$

$1 \times 1 = [1 \times p][p \times 1] + [1 \times 1]$

$$[1 \ \ x_{i1} \ \ x_{i2} \ \cdots] \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_p \end{bmatrix} = \beta_0 + \beta_1 x_{i1} \cdots$$

# Multiple linear regressoion

- Let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \mathbf{X} = n\left\{ \underbrace{\overbrace{\begin{bmatrix} 1 & x_{11} & \dots & x_{1p} \\ \vdots & \vdots & x_{ij} & \vdots \\ 1 & x_{n1} & \dots & x_{np} \end{bmatrix}}^{p+1}}_{\text{design matrix}} \right., \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \vdots \\ \beta_p \end{bmatrix}, \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

- Then we can write the model in a more compact form:

$$\mathbf{y}_{n\times 1} = \mathbf{X}_{n\times(p+1)}\boldsymbol{\beta}_{(p+1)\times 1} + \boldsymbol{\epsilon}_{n\times 1}$$

- **X** is called the *design matrix*

$$y = X\beta + \varepsilon$$

# Matrix notation

$I_{n\times n} = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{bmatrix}$

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

- $\epsilon$ is a random vector rather than a random variable

$Cov(\varepsilon_i, \varepsilon_j) = (i,j)^{th}$ entry of cov. matrix

- $E(\boldsymbol{\epsilon}) = 0$ and $Cov(\boldsymbol{\epsilon}) = \sigma^2 I$
- Note that $Cov$ means the "variance-covariance matrix"

$E(\vec{\varepsilon}) = 0 \Rightarrow E(\varepsilon_i) = 0$

# Mean, variance and covariance of a random vector

- Let $\mathbf{y}^T = [y_1, \ldots, y_n]$ be an $n$-component random vector. Then its mean and variance are defined as

$$\begin{aligned} E(\mathbf{y})^T &= [E(y_1), \ldots, E(y_n)] \\ Var(\mathbf{y}) &= E\left[(\mathbf{y} - E\mathbf{y})(\mathbf{y} - E\mathbf{y})^T\right] = E(\mathbf{y}\mathbf{y}^T) - (E\mathbf{y})(E\mathbf{y})^T \end{aligned}$$

- Let $\mathbf{y}$ and $\mathbf{z}$ be an $n$-component and an $m$-component random vector respectively. Then their covariance is an $n \times m$ matrix defined by

$$Cov(\mathbf{y}, \mathbf{z}) = E\left[(\mathbf{y} - E\mathbf{y})(\mathbf{z} - \mathbf{z})^T\right]$$

# Coming up next...

## Today we covered

- Motivation
- Assumptions
- Interpretation of $\beta$s
- More on confounding (omitted variable bias)
- Matrix notation for MLR

## Next time...

- estimation (more least squares)
- more detailed model diagnostics
- inference